

Phase Space Structures of k -threshold Sequential Dynamical Systems

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Abstract

Sequential dynamical systems (SDS) are used to model a wide range of processes occurring on graphs or networks. The dynamics of such discrete dynamical systems is completely encoded by their phase space, a directed graph whose vertices and edges represent all possible system configurations and transitions between configurations respectively.

Direct calculation of the phase space is in most cases a computationally demanding task. However, for some classes of SDS one can extract information on the connected component structure of phase space from the constituent elements of the SDS, such as its base graph and vertex functions.

We present a number of novel results about the connected component structure of the phase space for k -threshold dynamical system with binary state spaces. We establish relations between the structure of the components, the threshold value, and the update sequence. Also fixed-point reachability from garden of eden configurations is investigated and upper bounds for the length of paths in the phase space are shown to only depend on the size of the vertex set of the base graph.

1 Introduction

Graph dynamical systems (GDS) constitute a natural framework to model distributed systems such as biological networks or epidemics over social networks. Common to all types of GDS are their defining elements: a finite graph Y with vertex set $v[Y] = 1, 2, \dots, n$ and a finite set K whose elements x_v are associated to each vertex v of Y . In addition, the transition of each

vertex state at each time step depends on the states of its nearest neighbors through a *vertex function*. The actions of the single vertex functions are called *vertex updates*. A *system update* over the entire vertex set is determined by a scheme specifying a mechanism by which the individual vertex functions are sequentially applied so as to induce a dynamical system with map $F : K^n \rightarrow K^n$.

Originating from applications of computer science and theory of simulations, [2, 3, 4] *sequential dynamical systems* (SDS) generalize the concept of *cellular automata* (CA) [6, 5] by employing arbitrary finite graphs with no restrictions on the vertex functions (in contrast, CA are usually defined on lattices with one single vertex function) and most importantly by being characterized by an *asynchronous* update mechanism specified, for instance, by an *ordering* of the vertex set.

Applications of SDS include concurrent processes, distributed protocols [12], approximate discrete event simulations [7], transport simulations on irregular grids [9, 10], image treatment and reconstruction [8], and the modeling of the mechanisms of gene transcription in messenger RNA [11].

The *phase space* associated to a SDS map $F : K^n \rightarrow K^n$ is the finite directed graph with vertex set K^n and directed edges $(\mathbf{x}, F(\mathbf{x}))$. The structure of the phase space is governed by the properties of the base graph Y , the vertex functions f_v , and the update scheme. The phase space encodes the entire dynamics of the system by representing all system states and all possible transitions among them. The fundamental properties of SDS have been studied in [20, 21, 22, 23, 24, 12].

Understanding the long term dynamics of a SDS and being able to compare the evolution of different system configurations are vitally important tasks in many situations. However, the exhaustive calculation of the phase space is often resource demanding. Indeed, even for a sequential dynamical system over a relatively small base graph, the brute force calculation of periodic points and transient configurations is a computationally hard problem [12, 15, 1, 18, 19]. Research in SDS therefore focuses on deducing the properties of the phase space from the features of the constituents of the system. This analysis has a local to global character.

The purpose of the present article is to investigate the connected component structure (therefore unravelling the dynamics of the system completely) of k -threshold SDS with binary states over different classes of base graph. Threshold SDS arise naturally in the context of gene annotation methods such as functional linkage networks [16]. Furthermore, this class of graph dynamical systems is particularly well-suited for reachability studies [17] since they exclusively admit fixed points as periodic points.

We present a collection of results concerning the dependence of the component structure of different phase spaces on the value of the threshold and base graph. The base graphs we considered are complete graphs, circle

graphs, and two simple examples of tree: stars, and line graphs. A case of particular interest occurs when the threshold value is set to 2. In this scenario phase spaces consist of star-shaped components centered at a fixed point, and isolated fixed points, and this feature is shared by all four types of base graph. This special situation greatly facilitates the determination of basins of attractions and ω -limit sets.

In addition, we find upper bounds for the length of directed paths within connected components and, construct update orders for which paths of maximal length occur. Our upper bounds refine the results presented in [15] for the cases in question.

2 Preliminaries

Let Y be a simple, undirected graph (henceforth just a graph). Let $v[Y] = \{1, 2, \dots, n\}$ be the vertex set of Y and $e[Y]$ be its edge set.

Given $v \in v[Y]$ we use $n[v]$ to denote the set of vertices that are contained in the 1-neighborhood of v ordered increasingly. If $d(v)$ indicates the degree of v , then $|n[v]| = d(v) + 1$.

Let K be a finite set of states. To each vertex in $v[Y]$ we associate a *vertex state* $y_v \in K$. The n -tuple $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is called a *system state* or *system configuration*. The restriction of \mathbf{y} to the vertices of the set $n[v]$ is referred to as the *state neighborhood* of v and is written as $\mathbf{y}[v]$.

The state of each vertex is *updated* through a *vertex function* $f_v : K^{d(v)+1} \rightarrow K$ defined on $\mathbf{y}[v]$. Vertex functions can be incorporated into *Y -local functions* $F_Y : K^n \rightarrow K^n$ as

$$F_Y(y_1, y_2, \dots, y_n) = (y_1, y_2, \dots, y_{v-1}, f_v(n[v]), y_{v+1}, \dots, y_n). \quad (1)$$

We denote the family of Y -local functions over the vertices of Y by $\mathbf{F}_Y = (F_v)_{v \in v[Y]}$.

Let S_Y be the set of permutations (the symmetric group) over $v[Y]$.

Definition 2.1. Let Y be a graph, K be a finite set, $\pi = (v_{i_1}, v_{i_2}, \dots, v_{i_n}) \in S_Y$ be a permutation over $v[Y]$, and \mathbf{F}_Y the family of Y -local functions associated to the vertices of Y . We say that the triple (Y, \mathbf{F}_Y, π) is the permutation sequential dynamical system (which we simply refer to as SDS) over the base graph Y with state space K and update sequence π . The associated dynamical system map (SDS map) is $\mathbf{F}_\pi = K^n \rightarrow K^n$,

$$\mathbf{F}_\pi = F_{v_{i_n}} \circ F_{v_{i_{n-1}}} \circ \dots \circ F_{v_{i_1}}. \quad (2)$$

Consider a sequence of maps $(g_l)_{l=1}^n$ with $g_l : K^l \rightarrow K$. We say that the maps g_l *induce* the family of vertex functions $(f_v)_{v \in v[Y]}$ by setting $f_v = g_{d(v)+1}$. The SDS obtained from these vertex functions is called the SDS over Y *induced* by the sequence $(g_l)_{l=1}^n$.

Accordingly, in the case of regular graphs, an SDS is induced if all vertices of Y have identical vertex functions induced by g_l .

The *phase space* of an SDS map \mathbf{F}_π is the directed graph $\Gamma(\mathbf{F}_\pi)$ with vertex set K^n and edge set

$$e[\Gamma(\mathbf{F}_\pi)] = \{(\mathbf{y}, \mathbf{F}_\pi(\mathbf{y})) \mid \mathbf{y} \in K^n\}.$$

Starting from a system state \mathbf{x} , one can construct the following sequence

$$O^+(\mathbf{x}) = (\mathbf{x}, \mathbf{F}_\pi(\mathbf{x}), \mathbf{F}_\pi^2(\mathbf{x}), \mathbf{F}_\pi^3(\mathbf{x}), \dots) = (\mathbf{F}_\pi^k(\mathbf{x}))_{k=0}^\infty.$$

The sequence $O^+(\mathbf{x})$ is called the *forward orbit* (or a *time series*) of \mathbf{x} under \mathbf{F}_π . Since we only consider finite base graphs and vertex states are taken from a finite set, all forward orbits contain a finite number of distinct vertices. By definition, the orbit of a given system state traces out a directed walk through the phase space. The direction is from the initial vertex \mathbf{x} onward, that is, we have the alternating sequence of vertices and edges

$$(\mathbf{x}, \mathbf{x}\mathbf{F}_\pi(\mathbf{x}), \mathbf{F}_\pi(\mathbf{x}), \mathbf{F}_\pi(\mathbf{x})\mathbf{F}_\pi^2(\mathbf{x}), \mathbf{F}_\pi^2(\mathbf{x}), \dots).$$

Each of such walks is connected since it contains a path between any two of its vertices. Since any vertex of $\Gamma(\mathbf{F}_\pi)$ can be used as the initial point of an orbit, the phase space of an SDS map is the collection of all forward orbits, which its vertices generate.

Given a system state \mathbf{y} the *predecessors* of \mathbf{y} are those configurations \mathbf{x} such that $\mathbf{F}_\pi(\mathbf{x}) = \mathbf{y}$. A configuration that does not admit predecessors is called a *garden of Eden* (GOE). Such configurations can only occur as initial states of an SDS and can never be generated during its evolution.

As in the continuous case, one can recover the notions of periodic points and attractors for graph dynamical dynamical systems.

A *fixed point* of an SDS is a system state \mathbf{y} for which $\mathbf{F}_\pi(\mathbf{y}) = \mathbf{y}$. We say that an SDS *cycles* through a (finite) sequence of states $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_r)$ if $\mathbf{F}_\pi(\mathbf{y}_1) = \mathbf{y}_2$, $\mathbf{F}_\pi(\mathbf{y}_2) = \mathbf{y}_3$, \dots , $\mathbf{F}_\pi(\mathbf{y}_r) = \mathbf{y}_1$. The system states in $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_r)$ are called *periodic states*. Thus, a fixed point is a periodic cycle with one single system state.

Due to the fact that for finite base graphs (the only ones that we will consider in our study) the phase space of an SDS map possesses only a finite number of nodes and that each node of the phase space has out-degree 1 (SDS are deterministic), the forward orbit of any system state will eventually reach a periodic cycle [14].

Given an SDS map $\mathbf{F}_\pi : K^n \rightarrow K^n$, the set $M \subset K^n$ for which $\mathbf{F}_\pi(M) \subset M$ is called an *invariant set* of \mathbf{F}_π . An ω -**limit set** of a state of K^n is the set of periodic points of \mathbf{F}_π , $\mathbf{z} \in K^n$ such that $\mathbf{F}_\pi^m(\mathbf{y}) = \mathbf{z}$ for some $m \geq 0$.

The *basin of attraction* of an invariant set $M \subset K^n$ is the set of the points of K^n which possess ω -limit sets contained in M .

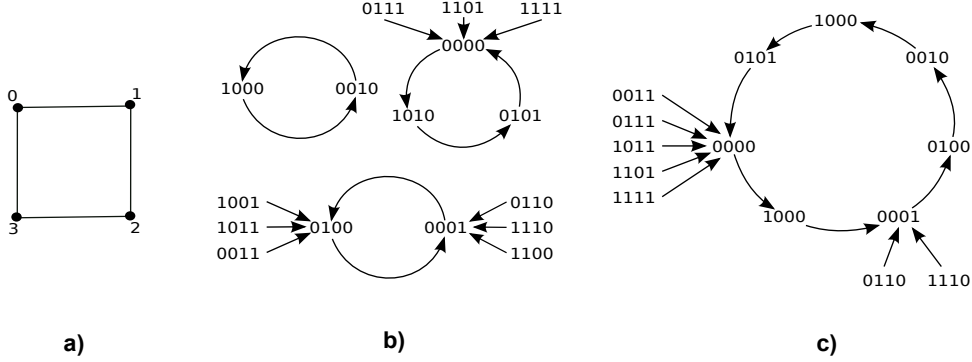


Figure 1: **a)** The graph Circ_4 . The phase spaces of the SDS generated by the vertex functions $\text{nor}_k(x_1, x_2, \dots, x_k) = (1 + x_1) \times (1 + x_2) \times \dots \times (1 + x_k)$ over Circ_4 for the update sequence $(0, 2, 1, 3)$ **(b)** and $(0, 1, 2, 3)$ **(c)**.

We set $\text{sum}_n : K^n \rightarrow \mathbb{N}^0$ to be

$$\text{sum}_q(y_1, y_2, \dots, y_q) = \sum_{i=1}^q y_i \quad (\text{computed in } \mathbb{N}^0).$$

Definition 2.2. For any integer k the k -simple threshold function $t^k : \mathbb{F}_2^q \rightarrow \mathbb{F}_2$ is defined by

$$t^k(\mathbf{y}) = \begin{cases} 0 & \text{if } \text{sum}_q(\mathbf{y}) < k \\ 1 & \text{otherwise} \end{cases}$$

We will use the term k -threshold SDS to refer to an SDS induced by k -simple threshold functions.

We exclusively consider k -threshold SDS with vertex states in the Boolean set \mathbb{F}_2 . To avoid repeated mentioning of their various constituents we will refer to this special type of k -threshold SDS as *binary k -threshold SDS*.

Any threshold SDS only admits fixed points as periodic points [12, 14]. This characteristic feature makes this class of SDS particular well-suited for reachability [17] and perturbation [14] studies.

The main goal of our analysis is to investigate the structure of the phase space of *binary k -threshold SDS* and to understand how the properties and size of their connected components vary as we change the base graph, the value of the threshold, and the update sequence. An estimation of the "length" of a connected components can be carried out through the *fixed point reachability* of its GOE configurations, that is, the length of the GOE-to-fixed point paths. In other words, given the SDS (Y, \mathbf{T}^k, π) we will

evaluate upper bounds and exact values of the smallest integer r such that $(\mathbf{T}_\pi^k)^r(\mathbf{y}) = (\mathbf{T}_\pi^k)^{r+1}(\mathbf{y})$ for the GOE \mathbf{y} [15].

We use the standard notation introduced in [12, 13] for the definition and vertex enumeration of *complete graphs* K_n , *star graphs* Star_{n-1} , *circle graphs* Circ_n , and *line graphs* Line_n over n vertices.

3 Phase Space Structures

In this section we investigate how the topology of the connected components of the phase space depends on the threshold value. The system states $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$ will play a special role in our investigation, and for sake of simplicity we will denote them by $\mathbf{0}$ and $\mathbf{1}$ respectively.

The trivial threshold $k = 0$ clearly makes little sense in the context of a binary state space (every possible system state is updated to $\mathbf{1}$ after just one application of the SDS-map for any base graph and any update sequence), and we will discard it altogether in our analysis.

In addition, we always assume that $1 \leq k \leq d(v) + 1$ where $d(v)$ is the greatest degree of a vertex in Y . In fact, if $k > d(v) + 1$ no vertex state can ever be updated to 1 and any system state is updated to $\mathbf{0}$ for any update order.

Under these assumptions, binary k -threshold SDS always admit at least one fixed point.

Lemma 3.1. *For a binary k -threshold SDS, $(Y, (T_v^k)_v, \pi)$ with $k \geq 1$, the state $\mathbf{0}$ is a fixed point of the SDS-map \mathbf{T}_π^k .*

Lemma 3.2. *For a binary k -threshold SDS, $(Y, (T_v^k)_v, \pi)$ the system state $\mathbf{1}$ is a fixed point of the SDS-map \mathbf{T}_π^k , if for all vertices $v \in v[Y]$, $d(v) + 1 \geq k$.*

We will treat the case of 2-threshold SDS separately. Not only is 2 the first nontrivial threshold value for the binary state space, it is also the value for which the phase spaces over the four types of base graph possess connected components with the same structure.

For the case of 1-threshold SDS we make the following observation.

Proposition 1. *After exactly one system update all the vertex states in a neighbourhood containing at least one 1 state when the update is applied, will be changed to 1. This happens independently of the update sequence.*

3.1 Complete Graphs

Complete graphs K_n offer the advantage of each vertex possessing a "global neighborhood", that is, it has all the others vertices in $v[K_n]$ as nearest neighbors. This feature simplifies considerably the analysis of the phase space of threshold SDS over complete graphs.

Theorem 3.3. *Let $[K_n, (T_v^k)_v, \pi]$ be a binary k -threshold SDS with $1 \leq k \leq n$. Then the phase space of \mathbf{T}_π^k consists of the union of 2 star-shaped connected components, whose centers are the fixed points $\mathbf{0}$ and $\mathbf{1}$. In addition, the structure of the phase space is independent of the update sequence π .*

Proof. Since every vertex of K_n belongs to a global neighborhood, the update of a vertex state affects the update of all the other vertex states. For a threshold function the update of x_i depends on $\text{sum}_n(\mathbf{x}[i])$, so clearly the order in which the vertices are updated is unimportant to the outcome of the sum.

Given an initial system state $\mathbf{x} \in \mathbb{F}_2^n$, the system will evolve in either of the following ways.

- If $\text{sum}_n(\mathbf{x}) < k$: the initial state contains less than k vertex 1 states, and each of the vertex states will be updated to 0. The update ends with the fixed point $\mathbf{0}$.
- If $\text{sum}_n(\mathbf{x}) \geq k$: the initial state contains more than k 1 states and all vertex states will be updated to 1. The update ends with the fixed point $\mathbf{1}$.

So the orbit of any system state is attracted either by $\mathbf{0}$ or $\mathbf{1}$ after one single iteration of the SDS-map. \square

Counting GOE configurations and determining the basins of attractions of $\mathbf{0}$ and $\mathbf{1}$ thereby become straightforward.

Corollary 1. *Let $[K_n, (T_v^k)_v, \pi]$ be a binary k -threshold SDS with $1 \leq k \leq n$. The basin of attraction of $\mathbf{0}$, $\mathcal{B}(\mathbf{0})$, consists of $\sum_{i=1}^{k-1} \binom{n}{i}$ points.*

Proof. We have $\binom{n}{i}$ system states that possess i vertex 1 states. To obtain the number of system states that end up at $\mathbf{0}$ after one update, we sum over i from 1 to $k-1$ to cover all the possible system states with the number of vertex 1 states ranging from 1 to $k-1$, that is, $|\mathcal{B}(\mathbf{0})| = \sum_{i=1}^{k-1} \binom{n}{i}$. \square

Corollary 2. *Let $[K_n, (T_v^k)_v, \pi]$ be a binary k -threshold SDS with $1 \leq k \leq n$. The basin of attraction $\mathcal{B}(\mathbf{1})$ of $\mathbf{1}$ consists of $2^n - \sum_{i=k}^{n-1} \binom{n}{i} - 2$ points.*

Proof. All the system states $\mathbf{x} \notin \mathcal{B}(\mathbf{0})$ that are not $\mathbf{0}$ or $\mathbf{1}$ belong to $\mathcal{B}(\mathbf{1})$. Thus, we subtract $|\mathcal{B}(\mathbf{0})| = \sum_{i=1}^{k-1} \binom{n}{i}$ from the total number of possible system states, 2^n , then we subtract 2 to account for the two fixed points, $\mathbf{0}$ and $\mathbf{1}$. We thus obtain $|\mathcal{B}(\mathbf{1})| = 2^n - \sum_{i=1}^{k-1} \binom{n}{i} - 2$. \square

3.2 Star_n

The center of a star graph occupies a special position, not only is it the only vertex with a global neighborhood, it is the only vertex common to all neighborhoods of the vertices in Star_n . With reference to the shape of a star, we introduce the following short-hand terminology. We refer to the elements of $v[\text{Star}_n]$ that are not the center as "arm vertices" and denote their states as x_a . The state of the center vertex is then called x_c .

For 1-threshold system one has

Theorem 3.4. *Let $[\text{Star}_n, (T_v^1)_v, \pi]$ be a binary 1-threshold SDS. Then the phase space of \mathbf{T}_π^1 consists of a directed tree rooted at the fixed point $\mathbf{1}$ for which the leaves have a distance at most 2 from the root, and the isolated fixed point $\mathbf{0}$.*

Proof. We observe that all the vertex states that are 1 or have neighboring vertices with state 1 will either stay 1 or be changed to 1. We have three cases depending on the state of the center and the 1 states in its neighborhood.

- $x_c = 0$ and $\text{sum}_{n+1}(\mathbf{x}[c]) = 0$ when x_c is updated.

This situation corresponds to the fixed point $\mathbf{0}$.

- $x_c = 0$ and $\text{sum}_{n+1}(\mathbf{x}[c]) \geq 1$ when x_c is updated.

The states x_a that are updated before x_c maintain their states since x_a is the only contribution to the sums $\text{sum}_2(\mathbf{x}[a])$. Then x_c is updated to 1 and so is every vertex state updated after x_c . Since now $x_c = 1$ all the states x_a that are not 1 already will get updated to 1 at the next system update. The system state will be attracted by the fixed point $\mathbf{1}$ after two systems updates at most.

- $x_c = 1$ when x_c is updated.

Every vertex state gets updated to 1 independently of the update sequence since x_c always equals 1. We reach the fixed point $\mathbf{1}$ after exactly one system update.

□

The following corollary is a consequence of the proof of theorem 3.4.

Corollary 3. *The SDS-map \mathbf{T}_π^1 possesses only two fixed points, the isolated fixed point $\mathbf{0}$ and the fixed point $\mathbf{1}$.*

Any system state of a binary 2-threshold SDS over Star_n is fully characterized by the position of the center vertex c in the update sequence π and by its state x_c . This observation allows us to devise four simple and very useful rules for 2-threshold SDS over star graphs.

Proposition 2. *Any state x_a will be updated to 0 when $x_c = 0$*

Proposition 3. *If $x_c = 1$, the states x_a maintain their values under vertex updates.*

Proposition 4. *To update x_c from 0 to 1, we must have at least two 1 states in x_a in $\mathbf{x}[c]$ when x_c is updated.*

Proposition 5. *To keep $x_c = 1$ upon update, $\mathbf{x}[c]$ must contain at least one state $x_a = 1$.*

Theorem 3.5. *Let $[Star_n, (T_v^2)_v, \pi]$ be a binary 2-threshold SDS. Then the phase space of \mathbf{T}_π^2 consists of star-shaped connected components centered at fixed points.*

Proof. The rules above allow us to fully characterize the phase space of \mathbf{T}_π^2 through the following five cases:

- $x_c = 0$ and there are less than two states $x_a = 1$ when x_c is updated.
If all states x_a are 0, by proposition 2, we have the fixed point $\mathbf{0}$. Otherwise there is one single $x_a = 1$. Independently of the sequence π , x_c can never be updated to 1 as only one vertex would contribute to sum_{n+1} . In addition, by proposition 2 the arm vertex with $x_a = 1$ will be updated to $x_a = 0$, while all the other vertices retain their 0 state. Since $x_c = 0$ before it is updated, all the arm vertices that were updated before c were necessarily updated to 0 according to proposition 2.
After one system update we therefore recover the fixed point $\mathbf{0}$.
- $x_c = 0$ and there are two or more states $x_a = 1$ when x_c is updated.
All the vertices that are updated before c will get the state 0 by proposition 2. Since $\text{sum}_{n+1}(\mathbf{x}[c]) \geq 2$, x_c gets updated to 1 (proposition 4). The states that were equal to 1 when x_c was updated will be left unchanged when they are updated (proposition 3). The next system update leaves the system state unchanged since $x_c = 1$ ensures that all the vertices updated before c maintain their state (proposition 3). Again $\mathbf{x}[c]$ contains more than two states equal to 1, leaving $x_c = 1$ (proposition 4), so that the states that are updated later also will not change.
- $x_c = 1$ and all the $x_a = 0$ when x_c is updated.
If all $x_a = 0$, they keep their 0 states (proposition 3) and x_c gets updated to 0 too, independently of the update sequence since $\text{sum}_{n+1}(\mathbf{x}[c]) = 1$. Again, we recover the fixed point $\mathbf{0}$.

- $x_c = 1$ and there is one $x_a = 1$ in $\mathbf{x}[c]$ when c is updated.

All arm states updated before c retain their initial value by proposition 3. The center state x_c stays 1 since $\text{sum}_{n+1}(\mathbf{x}[c]) = 2$. By proposition 3 all the arm states that are updated after c also maintain their values. Such configurations are therefore fixed points.

- $x_c = 1$ and there are two or more than two 1 states in $\mathbf{x}[c]$ when c is updated.

All the vertices updated before c maintain their original state and so does c when it is updated (propositions 3 and 5). The state $x_c = 1$ ensures that all the vertices updated after c keep their state (proposition 3). Such system state configurations are therefore fixed points.

We have thus proven that in each of these cases we have $(\mathbf{T}_\pi^2)^2(\mathbf{x}) = \mathbf{T}_\pi^2(\mathbf{x})$ for all system states $\mathbf{x} \in \mathbb{F}_2^{n+1}$. \square

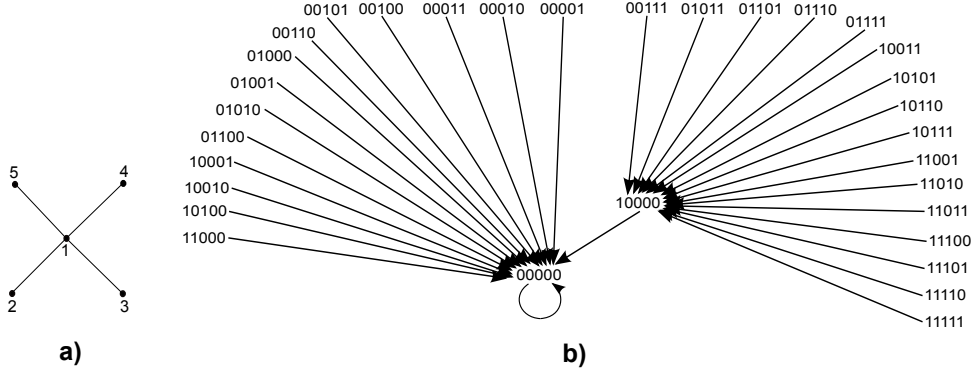


Figure 2: **a)** The graph $Star_4$. **b)** Phase space of the 3-threshold SDS over $Star_4$ with the identity update sequence.

Corollary 4. *The number of fixed points in the phase space of \mathbf{T}_π^k is $|\text{Fix}(\mathbf{T}_\pi^2)| = 2^n$.*

Proof. From the proof of theorem 3.5 we know that, apart from $\mathbf{0}$, the fixed points of \mathbf{T}_π^2 are characterized by $x_c = 1$ and by $\text{sum}_{n+1}(\mathbf{x}[c]) \geq 2$. There are n ways to construct a fix point with only one $x_a = 1$. Requiring two states x_a to be equal to 1 allows for the construction of $\binom{n}{2}$ fixed points. Therefore there are $\binom{n}{k}$ fixed points containing k vertex states $x_a = 1$. Counting $\mathbf{0}$, we have

$$|\text{Fix}(\mathbf{T}_\pi^2)| = 1 + \sum_{i=1}^n \binom{k}{i} = \sum_{i=0}^n \binom{k}{i} = 2^n.$$

\square

From the proof theorem 3.5, we see that n fixed points correspond to the configurations with one $x_a = 1$ and $x_c = 1$. We observe that the results of theorem 3.5 and corollary 4 hold for 2-threshold SDS only. For a generic k -threshold system with $k > 2$, the phase space consists of a single connected component, a tree rooted at the fixed point $\mathbf{0}$.

Theorem 3.6. *Let $[\text{Star}_n, (T_v^k)_v, \pi]$ be a binary k -threshold SDS with $2 < k \leq n + 1$. Then the phase space of \mathbf{T}_π^k consists of one single connected component, a tree rooted at $\mathbf{0}$ for which the leaves have a maximum distance of 2 from the root.*

Proof. Notice again that the state x_c and the position of the center vertex c in the update sequence π fully characterizes the phase space of \mathbf{T}_π^k . None of the vertices on the arms of the stars can ever be updated to $x_a = 1$ since $\text{sum}_2(\mathbf{x}[a]) \leq 2$ always, while we need to have at least k vertex 1 states in $\mathbf{x}[c]$ to update x_c to 1. We basically have two scenarios:

- $\text{sum}_{n+1}(\mathbf{x}[c]) < k$ when c is updated.

Then x_c is set to 0 along with all the other vertex states of the star graph. The update creates the fixed point $\mathbf{0}$. Such points have unit distance from $\mathbf{0}$.

- $\text{sum}_{n+1}(\mathbf{x}[c]) \geq k$ when c is updated.

During the system update, the state x_c is set to 1 while all the other vertex states become 0. The next update will set the system state to $\mathbf{0}$. The system passes through the state in which only $x_c = 1$ before reaching $\mathbf{0}$ upon a second update. Such points have distance 2 from $\mathbf{0}$.

We can thus conclude that the phase space of \mathbf{T}_π^k is constituted by a single tree-like (i.e. a directed tree with a loop at the root) connected component rooted at $\mathbf{0}$. The leaves have distance either 1 or 2 from the root. \square

From the proof above it follows immediately that

Corollary 5. *The system state $\mathbf{0}$ is the only fixed point for the map \mathbf{T}_π^k .*

3.3 Circ_n

Consider now 1-threshold SDS over circle graphs (or n -gons). We notice that (by proposition 1) all system state configurations that contain at least one vertex 1 state will eventually be updated to $\mathbf{1}$ upon iterative application of the SDS-map. We can thus conclude that all the system states but $\mathbf{0}$ belong to the same connected component rooted at the fixed point $\mathbf{1}$. As a consequence of proposition 1 for a vertex 1 state to be created after a

system update, at least one of its neighboring states must necessarily be 1 before the SDS-map is applied. System states that possess configurations consisting either of one single vertex 1 state or where all the 1 states are separated by one or more 0 state are therefore GOE configurations.

Lemma 3.7. *The phase space of a binary 1-threshold SDS over $Circ_n$ is constituted by the isolated fixed point $\mathbf{0}$ and a tree-like connected component rooted $\mathbf{1}$.*

With this in mind, we construct an update sequence for which the GOE-to-root path in the component containing $\mathbf{1}$ is the longest possible. We begin by considering a system state for which only the i -th vertex state is different from 0. Such configurations are the GOE with the greatest number of vertex 0 states that can be updated to 1. At least two vertex states must be updated to 1 at each application of the SDS-map (if only one state were updated to 1, we would have that a 0 neighboring a 1 is unaffected by the system update, thus contradicting proposition 1). The update order that minimizes the change in vertex states at each application of the SDS-map (thus achieving the longest GOE-to-root path) is necessarily one for which two 0 states are updated to 1 at each application of the SDS-map.

Starting from the vertex farthest away from i we update the vertex states on the sides of $(i + \lfloor n/2 \rfloor) \bmod n$ alternatively. We have the update sequence

$$\pi = ((i + \lfloor n/2 \rfloor) \bmod n, (i + \lfloor n/2 \rfloor - 1) \bmod n, (i + \lfloor n/2 \rfloor + 1) \bmod n, \\ (i + \lfloor n/2 \rfloor - 1) \bmod n, (i + \lfloor n/2 \rfloor + 2) \bmod n, \dots, i).$$

As the SDS map is applied iteratively more and more 1 states are added to the sides of x_i until the system reaches $\mathbf{1}$ after $\lfloor n/2 \rfloor$ updates.

Theorem 3.8. *Let $(Circ_n, (T_v^1)_v, \pi)$ be a binary 1-threshold SDS. There exists at least one update order $\tilde{\pi}$ for which the phase space of $T_{\tilde{\pi}}^2$ contains a path of length $\lfloor n/2 \rfloor$ and this is the maximum GOE-to-root path length attainable in the phase space.*

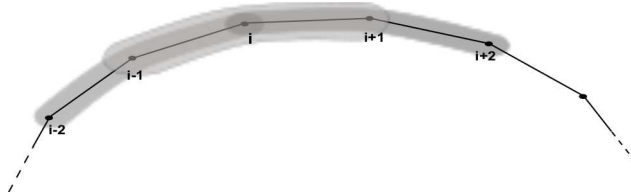


Figure 3: Compatible neighborhoods on $Circ_n$.

We continue by considering 2-threshold SDS over $Circ_n$.

Theorem 3.9. *Let $[\text{Circ}_n, (T_v^2)_v, \pi]$ be a binary 2-threshold SDS. Then the phase space of T_π^2 consists of star-shaped connected components centered at fixed points.*

Proof. By construction of Circ_n , for each vertex i the neighborhood state $\mathbf{x}[i] = (x_{i-1}, x_i, x_{i+1})$ has two overlapping neighborhoods $\mathbf{x}[i-1] = (x_{i-2}, x_{i-1}, x_i)$ and $\mathbf{x}[i+1] = (x_i, x_{i+1}, x_{i+2})$ with the identifications $n+1 = 1$ and $n+2 = 2$. We will refer to such set of neighborhoods as compatible neighborhoods. Compatible neighborhoods are all affected by the update of the vertex state x_i .

We will show that after one system update none of the vertex states in $\mathbf{x}[i]$ will change again when a second update is run.

We have two cases:

- Suppose that the Y -local function T_i^2 is applied to $\mathbf{x}[i]$ and x_i is updated to 1.

For x_i to be updated to 1, $\mathbf{x}[i]$ must have one of the following configurations at the time of the update: $(1, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ or $(0, 1, 1)$. After the Y -local function T_i^2 is applied we will have $(1, 1, 1)$, $(1, 1, 1)$, $(1, 1, 0)$ or $(0, 1, 1)$ respectively. These are all local fixed points of T_i^2 . In addition, the configurations $(1, 1, 1)$ and $(0, 1, 1)$ "lock" the update of x_{i+1} (if x_{i+1} were not updated before x_i) to 1 (by compatibility there are two 1 state in $\mathbf{x}[i+1]$). While the configurations $(1, 1, 1)$ and $(1, 1, 0)$ lock x_{i-1} to 1 in the same way.

In any of these cases there will always be at least two 1 states in $\mathbf{x}[i]$ when a second update is applied. Thus, the state x_i will remain fixed under further system updates and this holds for all the vertices of Circ_n .

- Suppose that the Y -local function T_i^2 is applied to $\mathbf{x}[i]$ and x_i is updated to 0.

For x_i to be updated to 0, $\mathbf{x}[i]$ must have one of the following configurations at the time of the update: $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$ or $(0, 0, 1)$. After the Y -local function T_i^2 is applied we will have $(0, 0, 0)$, $(0, 0, 0)$, $(1, 0, 0)$ or $(0, 0, 1)$ respectively. These are all local fixed points of T_i^2 . In addition, the configurations $(0, 0, 0)$ and $(1, 0, 0)$ "lock" the update of x_{i+1} (if x_{i+1} were not updated before x_i) to 0 (by compatibility there are two 0 state in $\mathbf{x}[i+1]$). While the configurations $(0, 0, 0)$ and $(0, 0, 1)$ lock x_{i-1} to 0 in the same way.

Any of these cases will ensure that there are at least two 0 states in $\mathbf{x}[i]$ when a second update is applied. Thus, the state x_i will remain fixed under further system updates and this holds for all the vertices of Circ_n .

Any given system state is therefore either a fixed point or will reach a fixed point after exactly one update. \square

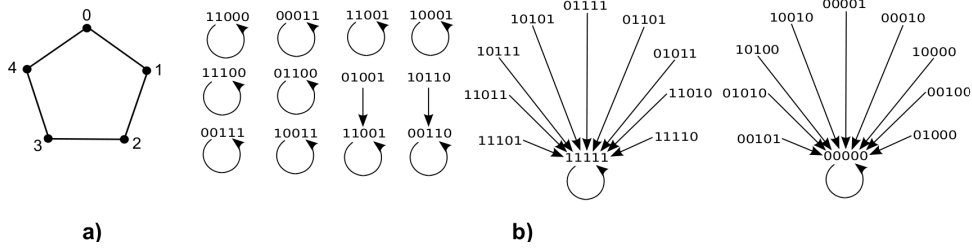


Figure 4: **a)** The graph $Circ_5$. **b)** Phase space of 2-threshold SDS with 5-gonal base graph and the identity update sequence.

Although n -gons are the simplest case of circle graphs, there is no immediate way to extract information about connected components or fixed points of a 2-threshold SDS by simple inspection of the system states. The most suitable method to obtain all the fixed points of the system is to construct a local-fixed-point graph as described in [12].

When we raise the threshold value to 3, we notice that for a vertex state $x_i = 1$ to remain 1 after the Y -local function is applied, all the states in $\mathbf{x}[i]$ must be 1 as well. Clearly, the fixed point $\mathbf{1}$ is the only system state for which this condition is satisfied for all vertices. If any of the vertex states in a system state is 0, the 0 vertex state will propagate within the system configuration until $\mathbf{0}$ is reached after a number of system updates. Graphically we have the opposite situation to the case of 1 threshold systems and an equivalent version of proposition 1 for which 0 and 1 vertex states are exchanged. It follows that for 3-threshold SDS over n -gons GOE configurations are those containing either one single 0 vertex state or where vertex 0 states are separated by one or more 1 states.

Lemma 3.10. *The phase space of a binary 3-threshold SDS over $Circ_n$ is constituted by the isolated fixed point $\mathbf{1}$ and a tree-like connected component directed towards the root $\mathbf{0}$.*

Considering the same update sequence constructed for theorem 3.8, with the difference that the starting GOE configuration has all vertex 1 states except for the one in the i^{th} position, we obtain the maximum path length in a connected component.

Theorem 3.11. *Let $(Circ_n, (T_v^3)_v, \pi)$ be a 3-threshold SDS. There exists at least one update order $\tilde{\pi}$ for which the phase space of \mathbf{T}_π^3 contains a path of*

length $\lfloor n/2 \rfloor$, and this is the maximum path length attainable in the phase space.

3.4 Line_n

If we set the threshold value to 1 for threshold SDS over line graphs, one recovers the same situation as for 1-threshold SDS over n -gons with the obvious difference in the neighborhoods of the end vertices. Again, we have that it is sufficient to have a single vertex 1 state to ensure convergence of the system state to $\mathbf{1}$ upon iterative update of the system for any update sequence as proposition 1 holds for any 1-threshold SDS. Analogously, GOE configurations will be those system state containing one single vertex 1 state or more vertex 1 states separated by at least one 0 state.

Theorem 3.12. *The phase space of a binary 1-threshold SDS over a line graph is constituted by a the isolated fixed point $\mathbf{0}$ and a tree-like connected component rooted at the fixed point $\mathbf{1}$.*

We can therefore apply the same arguments we have used for n -gons to find an update sequence that gives the longest GOE-to-root path in the connected component containing $\mathbf{1}$.

Starting at the GOE state $(0, \dots, 0, 1)$ we use the identity update sequence. It will take exactly $n - 1$ updates to reach the $\mathbf{1}$. This is the maximum number of applications of the SDS-map, since to produce at least n steps one should have an update order that does leave the entire system state unchanged under at least one update. Clearly this is only possible for fixed points.

Theorem 3.13. *There exists at least one update sequence that maximizes the GOE-to-root path length in the phase space of a binary 1-threshold SDS. The maximum length attainable is $n - 1$.*

Besides being important to numerous applications, line graphs constitute an example of how our results can be combined to work out the phase space component structure of new SDS from old ones. Theorem 3.5 and theorem 3.9 allow us to derive the following result.

Theorem 3.14. *Let $(\text{Line}_n, (T_v^2)_v, \pi)$ be a binary 2-threshold SDS. Then the phase space of \mathbf{T}_π^2 consists of star-shaped connected components centered at fixed points.*

Proof. First, one has to distinguish between two type of vertex present in a line graph. We have internal vertices, that is, all the vertices v with $d(v) = 2$ and the two leaves at the ends of the line with degree 1.

For the internal vertices we have the same neighborhood structure that we have seen in the proof of theorem 3.9. Thus, the result for 2-threshold SDS over Circ_n apply for internal vertices.

The update of the leaves depends only on the state of the internal vertex to which they are adjacent, in other words it behaves exactly as the update of an arm vertex in a star. Thus, theorem 3.5 ensures that the vertex states of the extremities are fixed after exactly one system update.

So for any vertex $v \in \text{Line}_n$, $\mathbf{x}[v]$ reaches a "local fixed point" after at most one application of T_v^2 .

We conclude that for any system state $(\mathbf{T}_\pi^2)^2(\mathbf{x}) = \mathbf{T}_\pi^2(\mathbf{x})$ holds. \square

Again, fixed points of line graphs can be determined through a generalization of the local fixed point method in [12]. The basins of attraction can be determined accordingly.

For a threshold value $k = 3$, we see immediately that the only fixed point is $\mathbf{0}$. In fact, all the systems states containing vertex 1 states will eventually be updated to $\mathbf{0}$ since there is always at least one neighborhood $\mathbf{x}[v]$ for which $\text{sum}_3 \mathbf{x}[v] < 3$. The GOE configurations for 3-threshold SDS over line graph will then have the same characteristics as those for 3-threshold SDS over n -gons.

To set up the update sequence that gives the longest possible GOE-to-root path length, we choose the state $\mathbf{1}$ as starting point. Since for any update sequence the end vertex states will necessarily be update to 0, we proceed from the ends towards the center vertices in the same fashion as we did for n -gons. We thus consider

$$\pi = (\lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1, \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor - 2, \lfloor n/2 \rfloor + 2, \dots, 1, n)$$

With this update sequence it takes $\mathbf{1}$ exactly $\lceil n/2 \rceil$ applications of the SDS-map to be updated to $\mathbf{0}$. Since for any two vertex 0 states that are separated by a string of 1 states on a line graph each system update must produce at least two new vertex 0 states. We have that the sequence that we constructed gives the longest GOE-to-root path (for the equivalent of proposition 1 to hold).

We collect our observations in the following theorem

Theorem 3.15. *The phase space of a binary 3-threshold SDS over Line_n is a single tree-like connected component rooted at the only fixed point, $\mathbf{0}$. In addition, there exists at least one update sequence that attains the upper bound of $\lceil n/2 \rceil$ to the GOE-to-root path length.*

4 Conclusions

In our survey of binary k -threshold SDS we have completely characterized the phase space of systems over complete, circle, line, and star graphs. We have proved that the reachability of fixed points from GOE configuration

Table 1: Maximal GOE-to-fixed-point path length for the four base graphs.

	K_n	Circ_n	Line_n	Star_n
$k = 1$	1	$\lfloor \frac{n}{2} \rfloor$	$n - 1$	2
$k = 2$	1	1	1	1
$k = 3$	1	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	2
$k > 3$	1	1	1	1

depends on the threshold value in all cases except for complete base graphs. We have explicitly constructed update sequences which maximize the GOE-to-root distance. This completely solves the problem of the reachability of fixed points [15, 1] from any given system state for the SDS in question. The work presented in this article suggests that a similar investigation can be extended to binary k -threshold SDS over generic tree graphs and that the GOE-to-root maximum length will again depend on the relation between the degree of non-leaf vertices and the threshold value. To obtain similar results for k -threshold SDS over regular graph would be also desirable. This especially important in the light of the results on the reduction of some classes of SDS to SDS induced by threshold and inverted threshold vertex functions over regular graphs presented in [15].

It would also be of interest to extend the investigation to state spaces containing more elements than \mathbb{F}_2 .

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